Covariant-Conics Decomposition of Quartics for 2D Object Recognition and Affine Alignment

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Abstract

This paper outlines a geometric parameterization of 2D curves where the parameterization is in terms of geometric invariants and terms that determine an intrinsic coordinate system. Thus, we present a new approach to handle two fundamental problems: single-computation alignment and recognition of 2D shapes under affine transformations. The approach is model-based, and every shape is first fit by an implicit fourth degree (quartic) polynomial. Based on the decomposition of this equation into three covariant conics, we are able to define a unique intrinsic reference system that incorporates usable alignment information contained in the implicit polynomial representation, a complete set of geometric invariants, and thus an associated canonical form for a quartic. This representation permits shape recognition based on 8 affine invariants. This is illustrated in experiments with real data sets.

1. Introduction

Implicit polynomial curves (i.e IP or algebraic curves) have proven to be powerful shape representations in model-based vision to handle the alignment and recognition of 2D shapes [2, 4]. To deal with affine transformations, we utilize the fact that 2D IPs can be decomposed into covariant conics. This means that when an affine transformation A is applied to the IP, every conic from the decomposition is transformed by A. The goal of the decomposition is to simplify the IP description by reduction to the well known case of conics. This implies an object-based canonical reference system (consisting of an intrinsic coordinate center and an intrinsic coordinate orientation) and a full set of geometric invariants.

Euclidean transformations can be similarly handled as a special affine case. More robust estimators and more detailed analysis of the Euclidean case is presented in [3] with the use of the complex representation of a 2D curve. In this paper, we focus our attention on affine transformations and quartics, but extensions of the approach to higher degrees can be developed.

In section 2, we outline how one can obtain quartic curves from a given raw data set by using a fitting algorithm. In section 3, we show how to decompose a quartic into three covariant conics. In section 4, we use our decomposition to obtain an affine canonical form for any quartic. Thus, affine alignment is processed in a single computation. We also exhibit a complete set of invariants under affine transformations with a natural distance measure on every invariant, which allows us to do recognition tasks in an efficient and optimal way. Finally we present some experimental results which illustrate robustness to noise and missing data in the estimation of invariants and our intrinsic reference system.

2. Shape Modeling with Quartics

A quartic is an algebraic curve of degree 4 in the plane, which is defined with Cartesian coordinates (x, y) by the IP equation:

$$\begin{aligned} f_4(x,y) = a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 \\ +a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy \\ +a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0 \end{aligned} \tag{1}$$

The number of polynomial coefficients $(a_{ij})_{0\leq i+j\leq 4}$ is 15. However, since the zero set of $f_4(x, y) = 0$ is unaffected by a multiplication of $f_4(x, y) = 0$ by a non-zero scalar, a quartic has 14 independent degrees of freedom. The *homogeneous* polynomial $H_4(x, y)$ of degree 4 in $f_4(x, y)$ is called the leading form of f_4. A conic, defined by a polynomial of degree 2 has 5 independent degrees of freedom.

There are different kinds of quartics depending on the number of real roots of $G(t) = y^{-4}H_4(x, y)$, $t = \frac{x}{y}$ (see section 3.1). There are two different types of non-singular conics: hyperbolas and ellipses. The leading form of a quartic can imply 4 asymptotes, 2 asymptotes, and no asymptotes in which case the quartic curve is bounded. For some quartics, a root in the leading form can be repeated, in which case it is called singular.

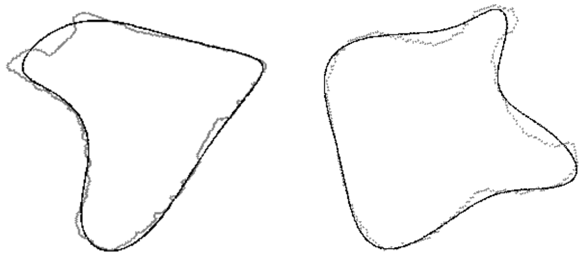

Figure 1. *The data set and its associated zero set obtained with the 3L IP fitting algorithm.*

To align and compare two shapes described by their boundaries, input data sets are assumed to be sets of points along boundaries (see the dotted points in Fig. 1). Before we can consider alignment and recognition, we must determine a quartic polynomial whose zero set approximates the data points. For this we use the 3-level (3L) fitting algorithm [1] which generates a robust and stable global representation. 3L fitting is numerically stable and repeatable, with respect to Euclidean transformations of the data set, and robust to noise and a moderate percentage of missing data. Fig. 1 illustrates the result of 3L fitting of two different objects by quartic IP curves (the solid lines).

3. Quartic Decomposition in Covariant Conics

Our aim in this section, as first formulated in [5], is to rewrite the polynomial function in (1) as the product of two conics, plus a third conic, namely:

$$f_4(x,y) = g_2(x,y)g_2'(x,y) + g_2''(x,y)$$

The proposed decomposition is unique for bounded curves and the three obtained conics are covariant under affine transformations.

3.1. The Leading Form

With the new variable $t = \frac{x}{y}$, the homogeneous leading form is rewritten as the fourth degree polynomial $a_{40}t^4 + a_{31}t^3 + a_{22}t^2 + a_{13}t + a_{40}$. This polynomial can always be factored as the product of two real second degree polynomials. Consequently:

$$\begin{array}{c} f_4(x,y) = a_{40}(x^2+\alpha_{11}xy+\alpha_{02}y^2)(x^2+\alpha_{11}'xy+\alpha_{02}'y^2) \\ +a_{30}x^3 + a_{21}x^2y + \cdots \end{array} \qquad (2)$$

where we have assumed that $a_{40} \neq 0$. Notice that the leading form decomposition is unique for polynomials with 2 or 0 asymptotes. For quartics with 4 asymptotes, the real roots can be coupled three different ways.

3.2. Third Degree Homogeneous terms

We now want to eliminate the third degree homogeneous terms by introducing linear terms $\alpha_{10}x + \alpha_{01}y$ and $\alpha_{10}'x + \alpha_{01}'y$ in each homogeneous quadratic factor in (2). After expansion, we observe that the coefficients of the third order terms are linear functions of α_{10}, α_{01}, α_{10}' and α_{01}'. Consequently, we can choose the values of these terms to eliminate all the third order terms by solving the following linear system:

$$a_{40}\begin{pmatrix} 1 & 0 & 1 & 0 \\ \alpha_{11}' & \alpha_{01}' & \alpha_{11} & \alpha_{01} \\ \alpha_{02}' & \alpha_{11}' & \alpha_{02} & \alpha_{11} \\ 0 & \alpha_{02}' & 0 & \alpha_{02} \end{pmatrix} \begin{pmatrix} \alpha_{10} \\ \alpha_{01} \\ \alpha_{10}' \\ \alpha_{01}' \end{pmatrix} = \begin{pmatrix} a_{30} \\ a_{21} \\ a_{12} \\ a_{03} \end{pmatrix} \qquad (3)$$

After this computation, the quartic polynomial is decomposed as:

$$\begin{array}{c} f_4(x,y) = a_{40}((x^2+\alpha_{11}xy+\alpha_{02}y^2+\alpha_{10}x+\alpha_{01}y) \\ (x^2+\alpha_{11}'xy+\alpha_{02}'y^2+\alpha_{10}'x+\alpha_{01}'y) + r_2(x,y)) \end{array} \qquad (4)$$

where the "remainder" $r_2(x,y)$ is of degree no more than two: $r_2(x,y) = b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{10}x + b_{01}y + b_{00}$.

Notice that the centers (t_x, t_y) and (t_x', t_y') of each conic factor g_2 and g_2', respectively, can be determined by solving:

$$\begin{pmatrix} 1 & \frac{\alpha_{11}}{2} \\ \frac{\alpha_{11}}{2} & \alpha_{02} \end{pmatrix} \begin{pmatrix} t_x \\ t_y \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \alpha_{10} \\ \alpha_{01} \end{pmatrix} \qquad (5)$$

since this computation is independent of the unknown constant term.

3.3. Uniqueness of the Decomposition

To express (4) as the product of two generic conics g_2 and g_2', we need to introduce constant terms α_{00} and α_{00}' into the two factors, respectively.

We recall that a quartic has 14 degrees of freedom, and a conic 5. We want to decompose the quartic as the product of two conics, plus a remainder term, which has $14 - 5 \times 2 = 4$ degrees of freedom. Consequently, the remainder can not be a fully independent conic. Due to the multiplicative factor on the remainder, g_2'' has only 3 degrees of freedom.

We propose an affine invariant constraint, which has the advantage of involving only linear computations (a non-linear alternative can be found in [6]). Indeed, we compute α_{00} and α_{00}' by constraining the remainder to be a conic with its center (c_x, c_y) at the mid-point of the line between points (t_x, t_y) and (t_x', t_y'). This constraint is linear because (5) is linear as a function of the conic coefficient when the center is given. Moreover, the coefficients of g_2'' involved in this constraint are linear functions of α_{00} and α_{00}'.

When an affine transformation is applied to a conic, its center is mapped by the same affine transformation, i.e the center of a conic is an affine covariant. Consequently, (c_x, c_y) is also a covariant of the two conic factors. This implies that our constraint on the third conic is invariant under affine transformations. With the uniqueness of the decomposition, this property ensures that g_2, g_2', and g_2'' are covariant conics with respect to the affine transformation applied to f_4.

Finally, after computation of α_{00} and α_{00}', we have decomposed every non-singular quartic as the product of two conics g_2 and g_2' plus a third conic g_2'', whose center is aligned with the centers of g_2 and g_2'. We call this the decomposition of a quartic into three covariant conics, hence a *covariant-conics decomposition*. Notice that for polynomials with 2 or 0 asymptotes, our decomposition is unique. For quartics with 4 asymptotes, 3 such decompositions exist.

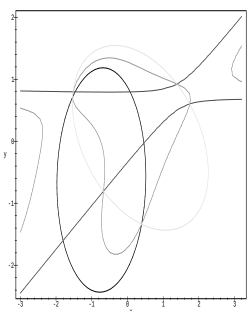

Figure 2. *The quartic of the hiking boot in Fig.1 is decomposed into three covariant conics (2 ellipses and 1 hyperbola).*

Fig. 2 illustrates the unique covariant-conics decomposition of an implicit polynomial fit to the hiking boot of Fig. 2, where one conic factor is an ellipse and the other a hyperbola.

4. Canonical Affine Form of Quartics

Since the conic decomposition is covariant, we have transformed the problems of alignment and recognition of quartics to the equivalent problems on a set of three conics. We will next show how to deduce an affine intrinsic reference system for a quartic curve, and then determine a complete set of affine invariants by computing the Euclidean invariants of the conics in the decomposition after transformation to a canonical reference system.

4.1. Affine Intrinsic Reference System of Quartics

There is no unique affine canonical form for a conic. An infinite number of affine transformations can change a given ellipse to a circle, for example. Nevertheless, and it is one of the main reasons for using quartics, it is possible to have an affine canonical form for a quartic by using the two covariant conics to define an affine intrinsic reference system.

A good candidate, for the origin of this reference system is (c_x, c_y) the center of g_2'' as described in the previous section. When the origin is known, we have to define an intrinsic reference system under linear transforms. First, we define $E = \begin{pmatrix} e_{20} & \frac{e_{11}}{2} \\ \frac{e_{11}}{2} & e_{02} \end{pmatrix}$ and $E' = \begin{pmatrix} e_{20}' & \frac{e_{11}'}{2} \\ \frac{e_{11}'}{2} & e_{02}' \end{pmatrix}$, the matrices associated with the leading terms of each conic g_2 and g_2'. These matrices are unique up to a scale factor. To define these matrices in a unique way, we assume that each conic is centered, and that its constant term is -1, before computing E and E'.

If a linear transform L is applied to the quartic, the previous two matrices become $L^t E L$ and $L^t E' L$, respectively, and the product $E'^{-1}E$ becomes $L^{-1}E'^{-1}EL$. We therefore deduce that the eigenvalues λ_i of $E'^{-1}E$ are affine invariants and, moreover, that the eigenvectors u_i of this matrix are linearly transformed by an affine transformation of the reference system. This eigenvector problem is equivalent to the following generalized eigenvector problem:

$$Eu_i = \lambda_i E' u_i \tag{6}$$

The two eigenvectors u_1 and u_2, which are solutions of (6), provide the directions of an affine intrinsic reference system for the quartic. However, this eigenvector problem does not always have a real solution. If the matrix E or E' is positive definite, i.e, if one of the conic factors is an ellipse, a real solution always exists. Indeed, when E' is positive definite, the square root B defined by $E' = BB^t$ exists. Consequently, the problem is equivalent to finding the eigenvectors $B^t u_i$ of the symmetric matrix $B^{-1}EB^{-t}$. Notice that in this case, the length of the vectors in the intrinsic coordinate system are also computed to transform g_2' to a unit circle.

When g_2 and g_2' are two hyperbolas, the previous properties do not apply. In this case, the leading term has 4 real roots, and the decomposition is not unique. However when roots are ordered, it is not difficult to show that real eigenvectors always exist if the two conic factors involve consecutive roots.

By diagonalizing the matrix $E'^{-1}E$, we have the direction of each axis of the intrinsic reference system. If we reverse the roles of E and E', the computed eigenvectors stay the same but the computed eigenvalues are inverted. Then, we order the axes by first choosing the axis which maximizes $\lambda_i + \frac{1}{\lambda_i}$, where λ_i is the associated eigenvalue. Moreover, if λ_1 is less than 1, we decide that the two ellipses are in reverse order. By convention, g_2 is defined as the largest conic factor. Then, both lengths of the vectors of the affine intrinsic reference system are computed to transform g_2' to a circle.

Therefore, we have defined the direction and the length of every axis of the affine intrinsic reference system. We now orient each axis through the center of the largest conic g_2 (see Fig. 3). After all these steps, the reference system is uniquely defined for quartics with zero or two asymptotes.

4.2. Affine Invariants of Quartics

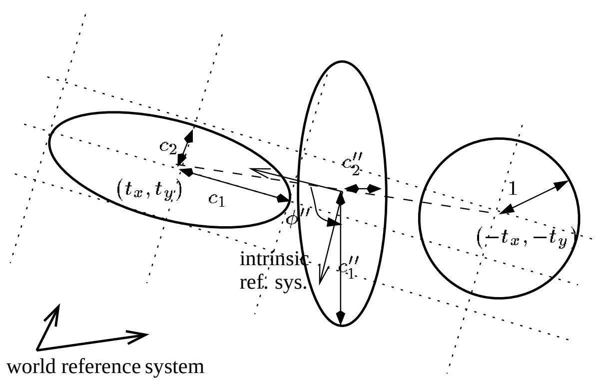

Figure 3. *An example of a complete affine invariants set of a quartic.*

When an intrinsic reference system has been found, we apply the inverse of the affine transformation to map the origin of the reference system to $(0, 0)$ and the coordinate orientations to orthogonal unit vectors. After this transformation, the Euclidean invariants of the set of 3 conics are affine invariants of the original quartic.

In this case, however, not all of the Euclidean invariants are useful. We want a largest independent subset. As shown in Fig. 3, a possible set of geometric affine invariants is:

- c_1, c_2: squared lengths of minor and major axes of the conic g_2 (positive for ellipses and negative for hyperbolas) in the intrinsic reference system,
- (t_x, t_y) the position of the center of the conic g_2,
- c_1'' and c_2'' the squared lengths of minor and major axis of the central conic g_2'',
- ϕ'' the angle of the major axis of g_2'',
- and the relative weight c between the conic factors and the central one.

The number of geometric invariants is then 8. An affine transformation has 6 degrees of freedom. The geometric invariant set is then complete, since $8 + 6 = 14$, which is equal to the number of quartic degrees of freedom.The invariants obtained are different from the classical algebraic invariants, because they are not restricted to being rational and polynomial functions of the polynomial coefficients, but rather are general functions involving roots and trigonometric functions. As a consequence, we call this new set of invariants *geometric invariants*. We want to emphasize the fact that the geometric interpretation of these invariants implies a natural distance to compare two sets, which is fundamental to practical object recognition.

The unique canonical form in the affine case is:

$$\begin{aligned} f_4(x, y) = (&\tfrac{(X_{intr.} - t_x)^2}{c_1} + \tfrac{(Y_{intr.} - t_y)^2}{c_2} - 1) \\ &((X_{intr.} + t_x)^2 + (Y_{intr.} + t_y)^2 - \epsilon) \\ &+ c(\tfrac{X''^2}{c_1''} + \tfrac{Y''^2}{c_2''} - 1) = 0 \end{aligned} \tag{7}$$

with $\epsilon = \pm 1$ and

$$\begin{pmatrix} X_{intr.} \\ Y_{intr.} \end{pmatrix} = \begin{pmatrix} \cos\phi'' & -\sin\phi'' \\ \sin\phi'' & \cos\phi'' \end{pmatrix} \begin{pmatrix} X'' \\ Y'' \end{pmatrix}$$

and where the intrinsic reference system is defined by:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_{1x} & u_{2x} \\ u_{1y} & u_{2y} \end{pmatrix} \begin{pmatrix} X_{intr.} \\ Y_{intr.} \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \end{pmatrix}$$

The vectors $u_1 = (u_{1x}, u_{1y})$ and $u_2 = (u_{2x}, u_{2y})$ are the two eigenvectors of (6).

4.3. Affine Alignment

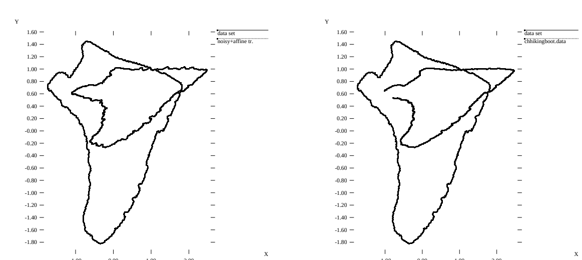

Figure 4. *Left, the original data set of the hiking boot and its perturbation with random Gaussian bumps of size 0.05 (object of size 3). Right, the original data and with 5% of the data missing. Then, in each case, an affine transformation is applied on the perturbed data set.*

We have computed the variances of alignment and invariance errors under affine transformations with additive blobby noise and partial occlusion, respectively (errors under Euclidean transformations for the same noise and partial occlusion are smaller). Fig. 4 is a typical example of noise or missing data combined with an affine transformation.

Table 1 is a summary of the relative errors obtained for the same two objects of Fig. 1. Statistics for each case are

	guitar	boot	guitar	boot
u_{1x}	6.2%	5.2%	12.2%	7.2%
u_{1y}	7.2%	6.4%	13.1%	9.3%
u_{2x}	4.2%	5.1%	5.8%	6.1%
u_{2y}	5.7%	80.9%	8.7%	70.0%
c_x	4.8%	3.4%	7.1%	6.1%
c_y	2.5%	1.8%	5.8%	3.1%

Table 1. *Percentage of error in the estimation of the affine intrinsic coordinate system under 0.05 noise and 5% missing data, respectively.*

computed from 50 different random realizations. We observed that the affine error is higher in comparison with the Euclidean case because the fitting is not affine invariant. For particular shapes instabilities can appears. Instability occurs for curves close to singular and for degenerate curves.

4.4. Recognition Tests

Table 2 presents the standard deviation relative to the value of every affine invariant under a 0.05 standard deviation noise and 5% of missing data, respectively. Due to the way we compute the affine invariants, by applying the inverse of the intrinsic reference system, the invariant robustness is related to the reference system robustness. The two last columns of Table 2 show how well separated the scatter invariants are under noise and missing data.

	guitar	boot	guitar	boot	noise	mis.
c_1	12.8%	10.8%	21.3%	17.3%	5.4	2.1
$\frac{c_2}{c_1}$	7.7%	13.6%	12.3%	49.7%	5.6	8.7
t_x	25.5%	16.2%	50.2%	32.3%	0.4	0.4
t_y	11.8%	5.1%	26.5%	6.0%	9.7	6.7
$\frac{c_1''}{c_1}$	11.1%	66.7%	16.2%	211%	11.3	13.6
$\frac{c_2''}{c_1''}$	4.4%	15.5%	5.3%	17.7%	21.6	15.8
ϕ''	36.4%	4.2%	39.4%	14.2%	35.6	40.3
c	13.8%	43.3%	28.1%	120%	16.4	15.9

Table 2. *Percentage of error in invariants under 0.05 noise and under 5% missing data, respectively. Two last columns show the ratio of the distance between the invariants of the guitar and the hiking boot over the std. dev. of the guitar, under noise and missing data, respectively.*

There exist several configurations of the three covariant conics that we have to handle as specific cases in the decomposition and alignment computations. For example, the conic factors have to be different than a circle. To define a unique reference system, the shape must be non-symmetric. Usually, it is possible to detect and handle these exceptions. So in practice, and to obtain a roughly robust algorithm, we must consider all these particular cases.

5. Conclusions

We have defined a unique intrinsic reference system and an associated canonical form for a quartic. This result is valid for most quartics, but not singular, degenerate or symmetric ones, for example, where the intrinsic reference system is not uniquely defined.

In some tests, under a transformation of shape, because of noise and missing data, the model curve obtained with the 3L fitting algorithm is not always decomposed as three conics of the same type for a shape and its transformation. In these situations, it is not possible to apply our approach, since the model of the original and perturbed shapes are of different mathematical types. An alternative model-based fitting algorithm is required for handling this problem. Another important improvement will be provided by an affine invariant fitting algorithm, since the 3L fitting algorithm used is only Euclidean invariant. More generally, a study of the accuracy of each geometric invariant related to the fitting process is needed.

In any case, the generalization of the given approach to compute a complete set of invariants and an intrinsic reference system for polynomials of higher degree than 4 is a very promising way to develop a generic framework and some powerful tools for model-based recognition and the alignment of 2D images.

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